Efficient operator-coarsening multigrid schemes for local discontinuous Galerkin methods

Robert Saye

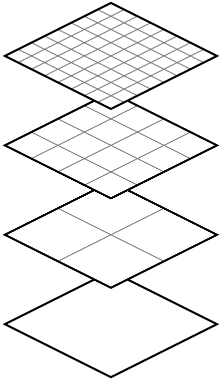
Dan Fortunato
Harvard

ICOSAHOM, July 9th 2018

Chris Rycroft

Introduction

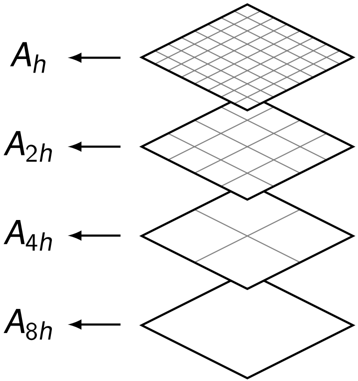
hp-multigrid works well for CG, but can perform poorly for DG... **why?**



Introduction

hp-multigrid works well for CG, but can perform poorly for DG... **why?**

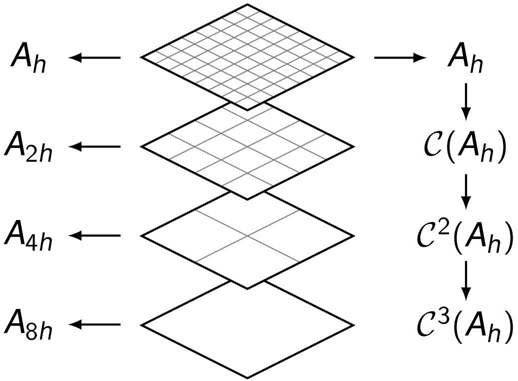
1. Discretize on hierarchy of meshes



Introduction

hp-multigrid works well for CG, but can perform poorly for DG... **why?**

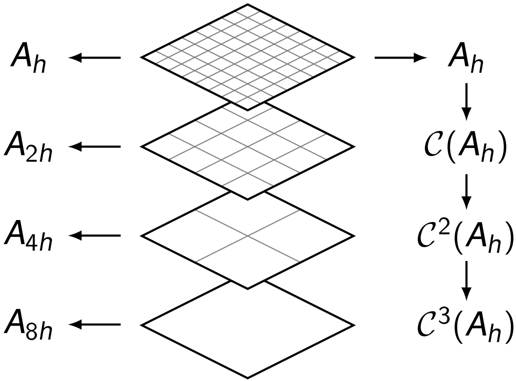
1. Discretize on hierarchy of meshes
2. Automatically build coarse operators: $\mathcal{C}(A_h) \to R_h^{2h} A_h I_{2h}^h$



Introduction

hp-multigrid works well for CG, but can perform poorly for DG... **why?**

1. Discretize on hierarchy of meshes
2. Automatically build coarse operators: $\mathcal{C}(A_h) \to R_h^{2h} A_h I_{2h}^h$

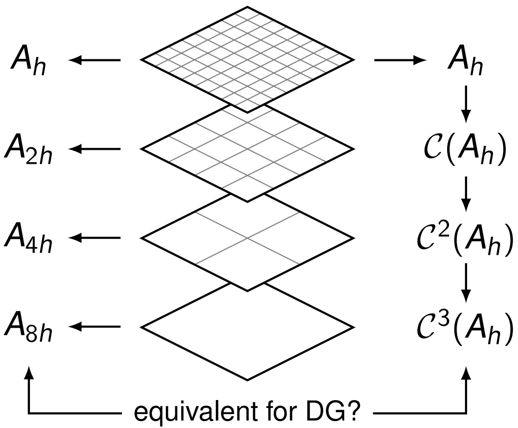


Avoids need to build coarse:
- ✓ quadrature rules
- ✓ lifting operators
- ✓ face-to-element maps
- ✓ mesh-related plumbing

Introduction

hp-multigrid works well for CG, but can perform poorly for DG... **why?**

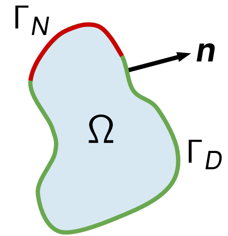
1. Discretize on hierarchy of meshes
2. Automatically build coarse operators: $\mathcal{C}(A_h) \to R_h^{2h} A_h I_{2h}^h$



Avoids need to build coarse:
- ✓ quadrature rules
- ✓ lifting operators
- ✓ face-to-element maps
- ✓ mesh-related plumbing

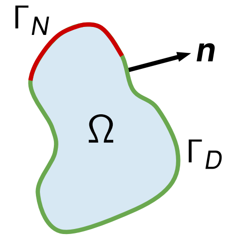
Consider the Poisson problem

$$
\begin{aligned}
-\nabla^2 u &= f \quad \text{in } \Omega \\
u &= g \quad \text{on } \Gamma_D \\
\nabla u \cdot \boldsymbol{n} &= h \quad \text{on } \Gamma_N
\end{aligned}
$$



DG for elliptic problems
General approach

Consider the Poisson problem



$$
\begin{aligned}
-\nabla^2 u &= f && \text{in } \Omega \\
u &= g && \text{on } \Gamma_D \\
\nabla u \cdot \boldsymbol{n} &= h && \text{on } \Gamma_N
\end{aligned}
$$

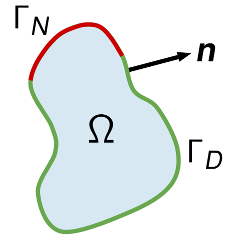
$$\downarrow$$

$$
\begin{aligned}
\boldsymbol{q} &= \nabla u && \text{in } \Omega \\
-\nabla \cdot \boldsymbol{q} &= f && \text{in } \Omega \\
u &= g && \text{on } \Gamma_D \\
\boldsymbol{q} \cdot \boldsymbol{n} &= h && \text{on } \Gamma_N
\end{aligned}
$$

DG for elliptic problems
General approach

Consider the Poisson problem

$$
\begin{aligned}
-\nabla^2 u &= f \quad \text{in } \Omega \\
u &= g \quad \text{on } \Gamma_D \\
\nabla u \cdot \boldsymbol{n} &= h \quad \text{on } \Gamma_N
\end{aligned}
$$



$$\downarrow$$

$$
\begin{aligned}
\boldsymbol{q} &= \nabla u \quad \text{in } \Omega \\
-\nabla \cdot \boldsymbol{q} &= f \quad \text{in } \Omega \\
u &= g \quad \text{on } \Gamma_D \\
\boldsymbol{q} \cdot \boldsymbol{n} &= h \quad \text{on } \Gamma_N
\end{aligned}
$$

Elements are coupled via **numerical fluxes**.

The local discontinuous Galerkin method
A popular choice

- Numerical fluxes “upwind” $\boldsymbol{q}$ and u in opposite directions

The local discontinuous Galerkin method

A popular choice

- Numerical fluxes "upwind" $\boldsymbol{q}$ and u in opposite directions
- Leads to a system of the form

$$
\begin{aligned}
&\text{“}\boldsymbol{q} = \nabla u\text{”} && \int_\Omega \boldsymbol{q}_h \cdot \boldsymbol{w}_h - \int_\Omega G u_h \cdot \boldsymbol{w}_h = \int_{\Gamma_D} g\, \boldsymbol{w}_h^- \cdot \boldsymbol{n} \\
&\text{“}-\nabla \cdot \boldsymbol{q} = f\text{”} && \int_\Omega \boldsymbol{q}_h \cdot G v_h + \text{penalty} = \int_\Omega f\, v_h + \int_{\Gamma_N} h\, v_h^-
\end{aligned}
$$

G is the **discrete gradient operator**:

$$G = \text{broken gradient} + \text{lifting operator}$$

where the lifting operator accounts for jumps between elements.

The local discontinuous Galerkin method
A popular choice

- Numerical fluxes "upwind" $\boldsymbol{q}$ and u in opposite directions
- Leads to a system of the form

$$
\begin{aligned}
m(\boldsymbol{q}_h, \boldsymbol{w}_h) - \mathrm{grad}(u_h, \boldsymbol{w}_h) &= j(\boldsymbol{w}_h) \\
-\mathrm{div}(\boldsymbol{q}_h, v_h) + \tau(u_h, v_h) &= k(v_h)
\end{aligned}
$$

The local discontinuous Galerkin method

A popular choice

- Numerical fluxes "upwind" $\boldsymbol{q}$ and u in opposite directions
- Leads to a system of the form

mass

$$
\begin{aligned}
m(\boldsymbol{q}_h, \boldsymbol{w}_h) - \mathrm{grad}(u_h, \boldsymbol{w}_h) &= j(\boldsymbol{w}_h) \\
-\,\mathrm{div}(\boldsymbol{q}_h, v_h) + \tau(u_h, v_h) &= k(v_h)
\end{aligned}
$$

The local discontinuous Galerkin method

A popular choice

- Numerical fluxes "upwind" $\boldsymbol{q}$ and u in opposite directions
- Leads to a system of the form

mass → m; gradient → grad

$$
\begin{aligned}
m(\boldsymbol{q}_h, \boldsymbol{w}_h) - \operatorname{grad}(u_h, \boldsymbol{w}_h) &= j(\boldsymbol{w}_h) \\
-\operatorname{div}(\boldsymbol{q}_h, v_h) + \tau(u_h, v_h) &= k(v_h)
\end{aligned}
$$

The local discontinuous Galerkin method

A popular choice

- Numerical fluxes "upwind" $\boldsymbol{q}$ and u in opposite directions
- Leads to a system of the form

mass → ; gradient → ; divergence →

$$
\begin{aligned}
m(\boldsymbol{q}_h, \boldsymbol{w}_h) - \mathrm{grad}(u_h, \boldsymbol{w}_h) &= j(\boldsymbol{w}_h) \\
-\,\mathrm{div}(\boldsymbol{q}_h, v_h) + \tau(u_h, v_h) &= k(v_h)
\end{aligned}
$$

The local discontinuous Galerkin method

A popular choice

- Numerical fluxes "upwind" $\boldsymbol{q}$ and u in opposite directions
- Leads to a system of the form

mass — gradient

$$m(\boldsymbol{q}_h, \boldsymbol{w}_h) - \text{grad}(u_h, \boldsymbol{w}_h) = j(\boldsymbol{w}_h)$$

$$-\,\text{div}(\boldsymbol{q}_h, v_h) + \tau(u_h, v_h) = k(v_h)$$

divergence — penalty

The local discontinuous Galerkin method

A popular choice

- Numerical fluxes "upwind" $\boldsymbol{q}$ and u in opposite directions
- Leads to a system of the form

mass, gradient

$$
\begin{aligned}
m(\boldsymbol{q}_h, \boldsymbol{w}_h) - \mathrm{grad}(u_h, \boldsymbol{w}_h) &= j(\boldsymbol{w}_h) \\
-\,\mathrm{div}(\boldsymbol{q}_h, v_h) + \tau(u_h, v_h) &= k(v_h)
\end{aligned}
$$

RHS + BCs

divergence, penalty

Primal formulation

"Eliminate-then-discretize"

Eliminate $\boldsymbol{q}_h$ to get bilinear form of Laplacian: $a(u_h, v_h) = \ell(v_h)$

$$a(u_h, v_h) = \int_\Omega G u_h \cdot G v_h + \text{penalty}$$

$$\ell(v_h) = \int_\Omega f\, v_h - \int_{\Gamma_D} g\, G v_h^- \cdot \boldsymbol{n} + \int_{\Gamma_N} h\, v_h^-$$

Primal formulation

"Eliminate-then-discretize"

Eliminate $\boldsymbol{q}_h$ to get bilinear form of Laplacian: $a(u_h, v_h) = \ell(v_h)$

$$a(u_h, v_h) = \int_\Omega G u_h \cdot G v_h + \text{penalty}$$
$$\ell(v_h) = \int_\Omega f\, v_h - \int_{\Gamma_D} g\, G v_h^- \cdot \boldsymbol{n} + \int_{\Gamma_N} h\, v_h^-$$

Discretization leads to symmetric positive (semi)definite system

$$A_h u_h = \ell_h.$$

$A_h = -M_h \Delta_h$ and Δ_h is **discrete Laplacian operator**,

$$\Delta_h = D_h G_h + T_h.$$

$$\begin{pmatrix} M_h = \text{mass matrix} & G_h = \text{gradient matrix} \\ T_h = \text{penalty matrix} & D_h = \text{divergence matrix} = -M_h^{-1} G_h^\top M_h \end{pmatrix}$$

Flux formulation

"Discretize-then-eliminate"

$$
\begin{aligned}
m(\boldsymbol{q}_h, \boldsymbol{w}_h) - \operatorname{grad}(u_h, \boldsymbol{w}_h) &= j(\boldsymbol{w}_h) \\
-\operatorname{div}(\boldsymbol{q}_h, v_h) + \tau(u_h, v_h) &= k(v_h)
\end{aligned}
$$

Flux formulation

“Discretize-then-eliminate”

$$\begin{bmatrix} M_h & -M_h G_h \\ -M_h D_h & M_h T_h \end{bmatrix} \begin{bmatrix} \boldsymbol{q}_h \\ u_h \end{bmatrix} = \begin{bmatrix} j_h \\ k_h \end{bmatrix}$$

$$\begin{aligned} M_h &= \text{mass matrix} \\ G_h &= \text{gradient matrix} \\ D_h &= \text{divergence matrix} = -M_h^{-1} G_h^\top M_h \\ T_h &= \text{penalty matrix} \end{aligned}$$

Flux formulation
"Discretize-then-eliminate"

$$\begin{bmatrix} M_h & -M_h G_h \\ -M_h D_h & M_h T_h \end{bmatrix} \begin{bmatrix} \boldsymbol{q}_h \\ u_h \end{bmatrix} = \begin{bmatrix} j_h \\ k_h \end{bmatrix}$$

$$\begin{aligned} M_h &= \text{mass matrix} \\ G_h &= \text{gradient matrix} \\ D_h &= \text{divergence matrix} = -M_h^{-1} G_h^\top M_h \\ T_h &= \text{penalty matrix} \end{aligned}$$

M_h is block diagonal, so can easily take Schur complement:

$$A_h u_h = \ell_h$$

where $A_h = -M_h \Delta_h$ and $\Delta_h = D_h G_h + T_h$.

Primal formulation

"Eliminate-then-discretize"

$$-D_h G_h + T_h$$

$$\downarrow$$

$$\mathcal{C}(-D_h G_h + T_h)$$

Flux formulation

"Discretize-then-eliminate"

$$\begin{bmatrix} I & -G_h \\ -D_h & T_h \end{bmatrix}$$

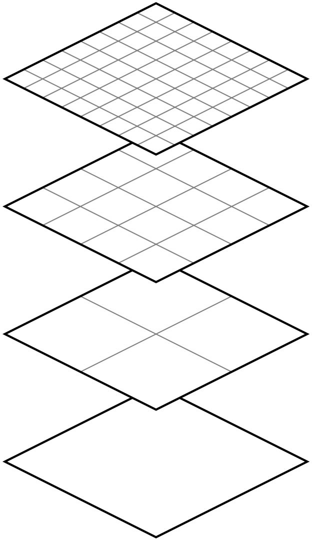
$$\downarrow$$

$$\begin{bmatrix} I & -\mathcal{C}(G_h) \\ -\mathcal{C}(D_h) & \mathcal{C}(T_h) \end{bmatrix}$$

$$\downarrow$$

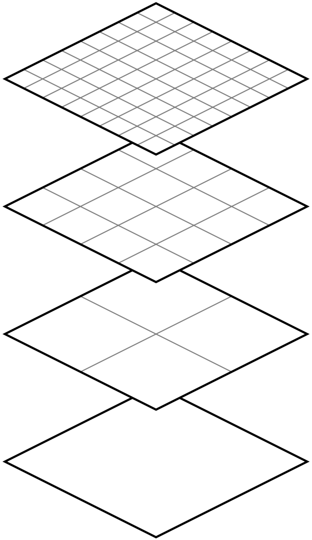
$$-\mathcal{C}(D_h)\mathcal{C}(G_h) + \mathcal{C}(T_h)$$

Multigrid
Coarse operators



Multigrid
Coarse operators

"geometric"



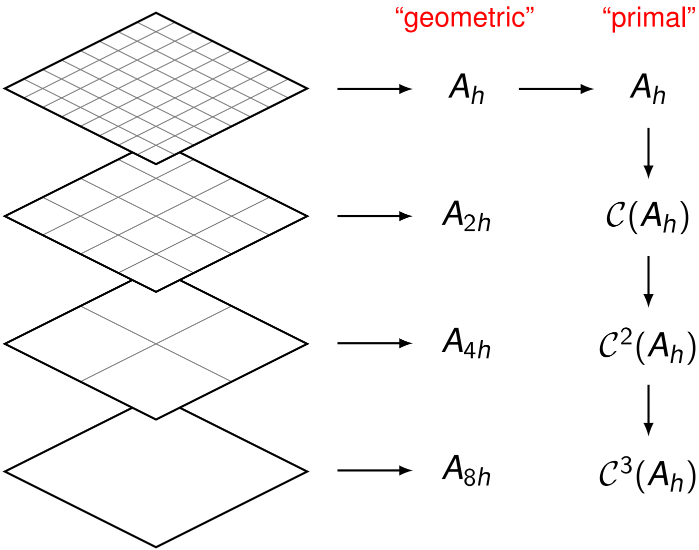
$\longrightarrow A_h$

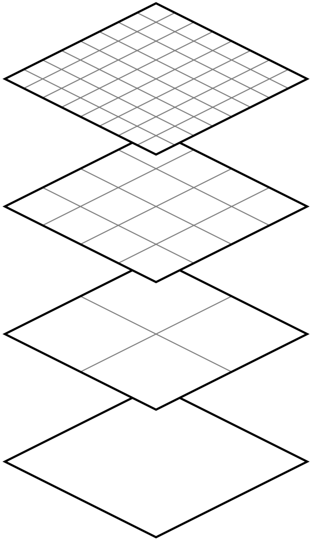
$\longrightarrow A_{2h}$

$\longrightarrow A_{4h}$

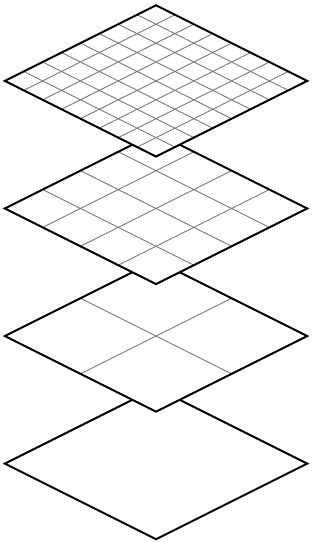
$\longrightarrow A_{8h}$

Multigrid
Coarse operators





"geometric"	"primal"	"flux"
A_h	A_h	$-D_h G_h + T_h$
A_{2h}	$\mathcal{C}(A_h)$	$-\mathcal{C}(D_h)\mathcal{C}(G_h) + \mathcal{C}(T_h)$
A_{4h}	$\mathcal{C}^2(A_h)$	$-\mathcal{C}^2(D_h)\mathcal{C}^2(G_h) + \mathcal{C}^2(T_h)$
A_{8h}	$\mathcal{C}^3(A_h)$	$-\mathcal{C}^3(D_h)\mathcal{C}^3(G_h) + \mathcal{C}^3(T_h)$

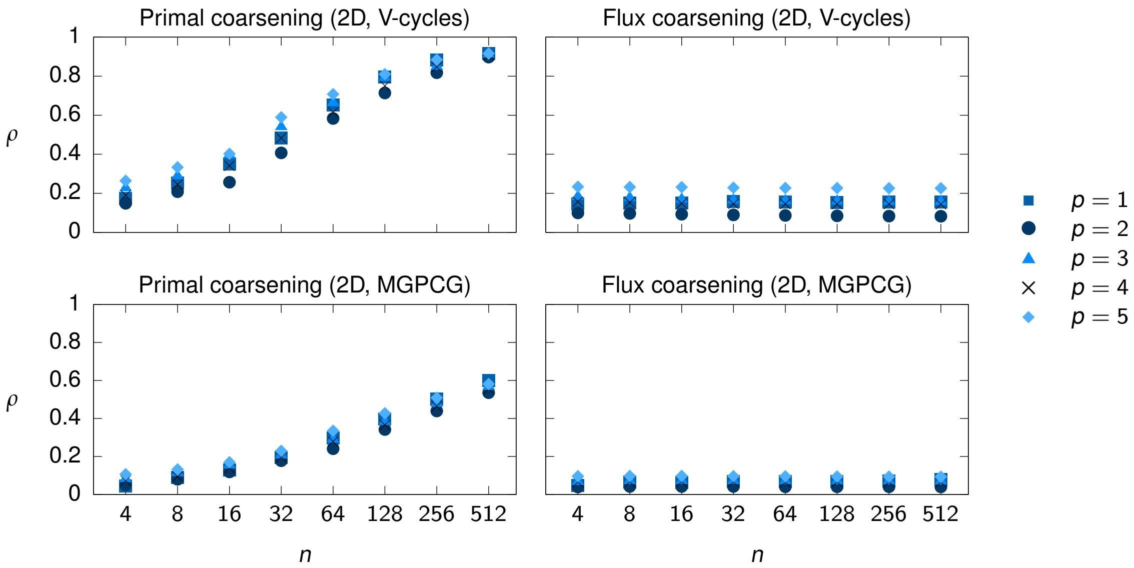


	"geometric"	"primal"	"flux"
→	A_h →	A_h →	$-D_h G_h + T_h$
→	A_{2h}	$\mathcal{C}(A_h)$	$-\mathcal{C}(D_h)\mathcal{C}(G_h) + \mathcal{C}(T_h)$
→	A_{4h}	$\mathcal{C}^2(A_h)$	$-\mathcal{C}^2(D_h)\mathcal{C}^2(G_h) + \mathcal{C}^2(T_h)$
→	A_{8h}	$\mathcal{C}^3(A_h)$	$-\mathcal{C}^3(D_h)\mathcal{C}^3(G_h) + \mathcal{C}^3(T_h)$

A_{8h} ← equivalent! → $-\mathcal{C}^3(D_h)\mathcal{C}^3(G_h) + \mathcal{C}^3(T_h)$

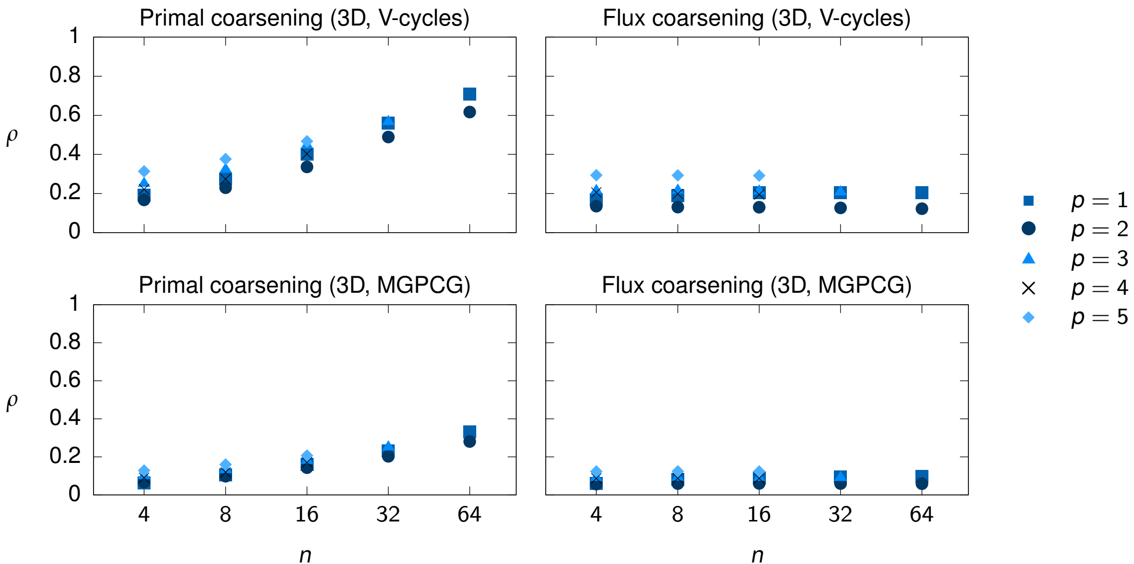
Example

Uniform Cartesian grids, 2D



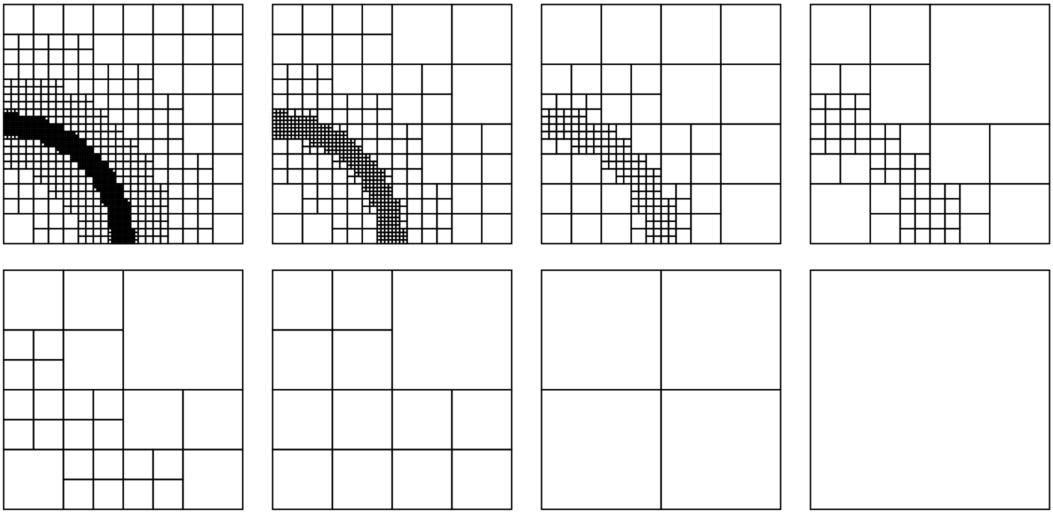
Example

Uniform Cartesian grids, 3D

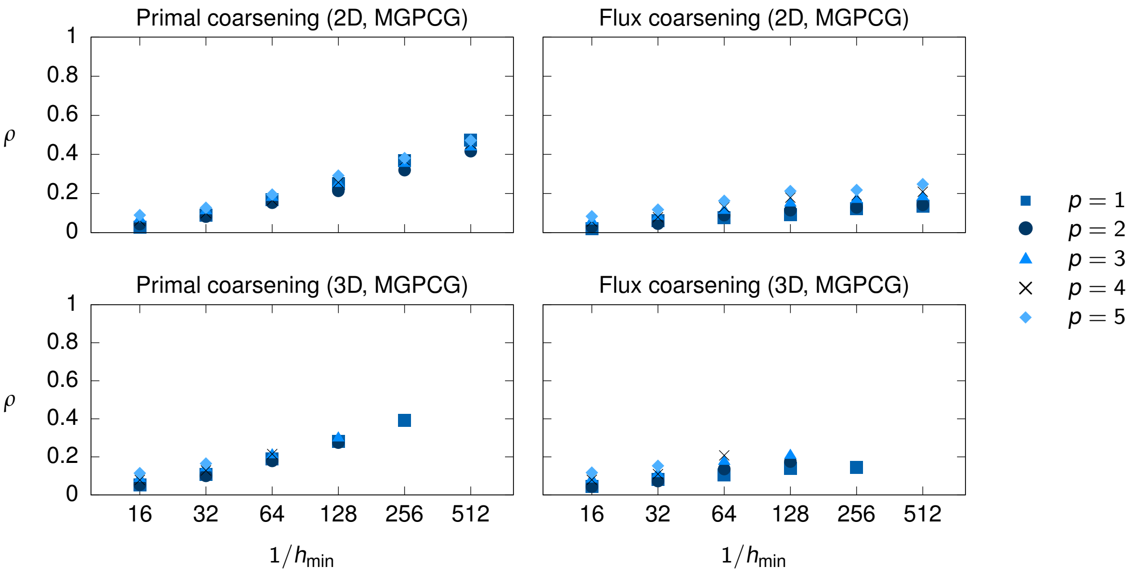


Example

Adaptive grids

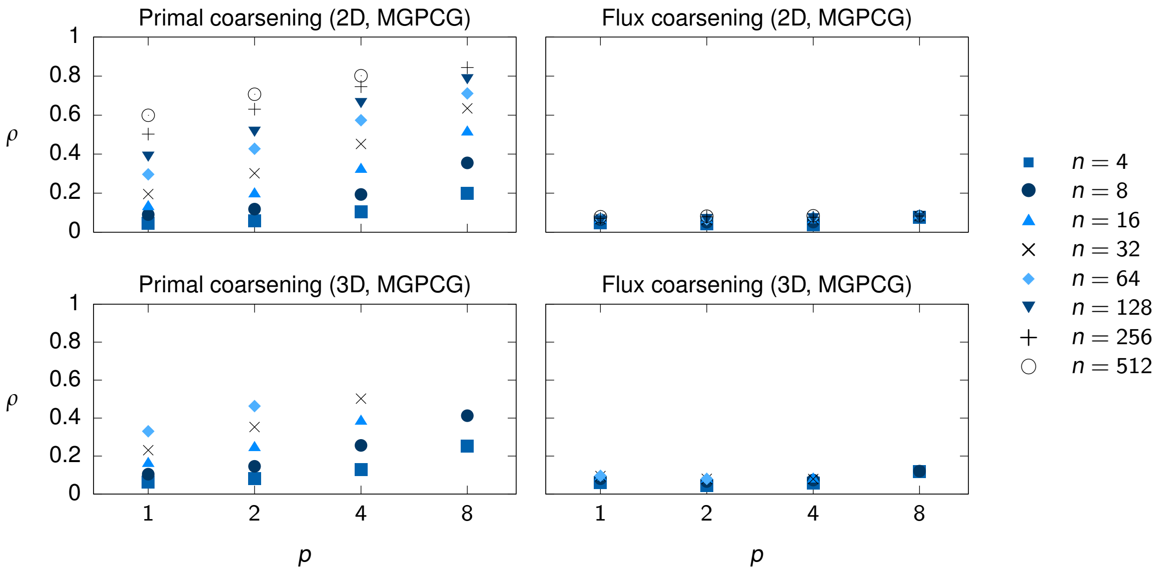


Example
Adaptive grids



Example

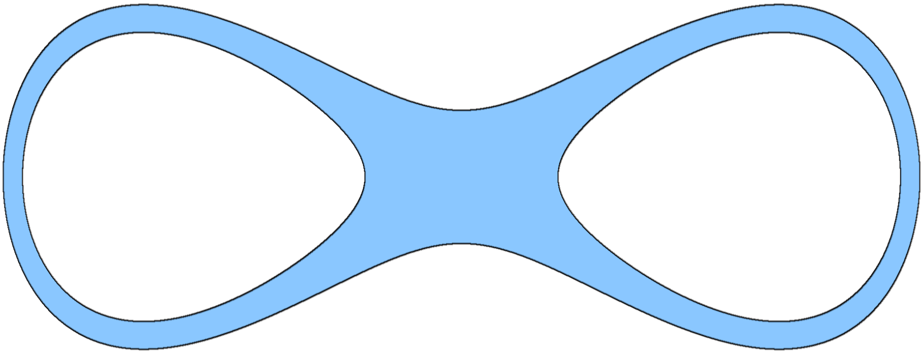
p-multigrid: $p \to p/2 \to \cdots \to 1$



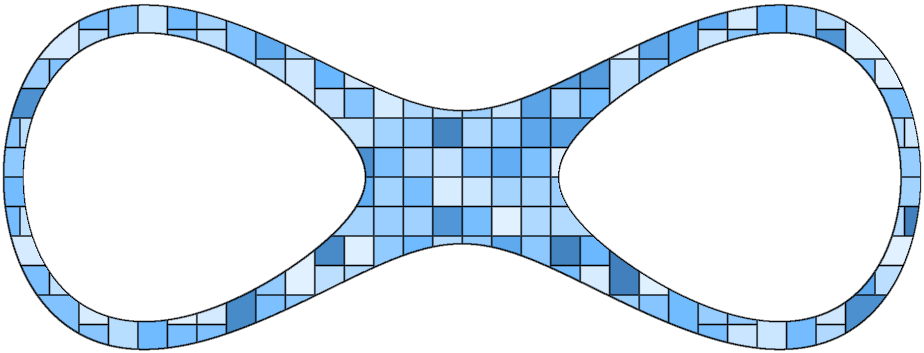
Example

Implicitly defined mesh — single phase

$$-\nabla^2 u = 0, \quad u|_\Gamma = 0$$

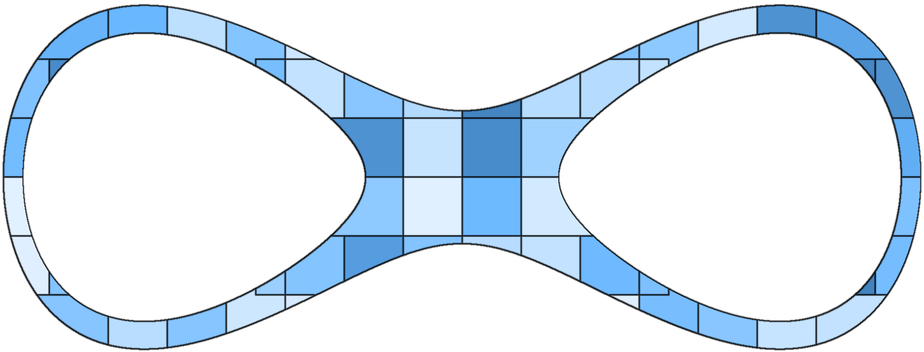


Example
Implicitly defined mesh — single phase



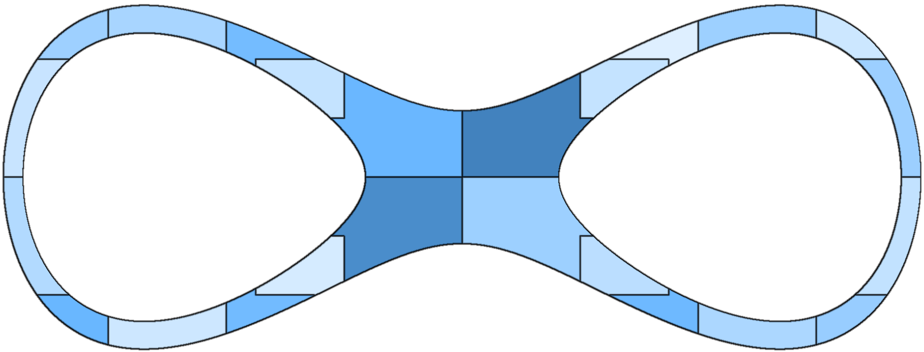
Example

Implicitly defined mesh — single phase



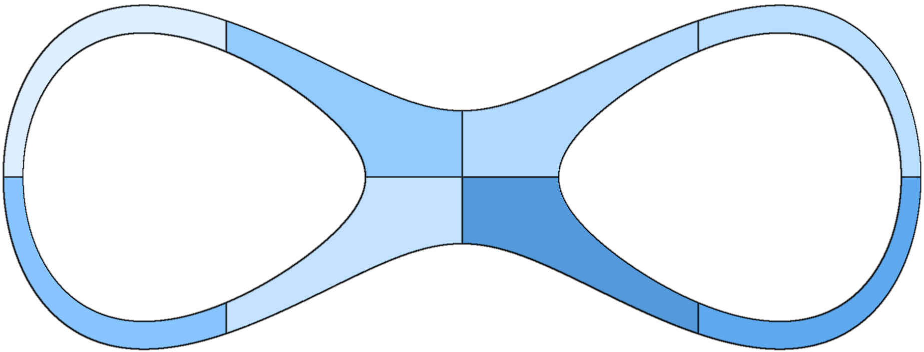
Example

Implicitly defined mesh — single phase



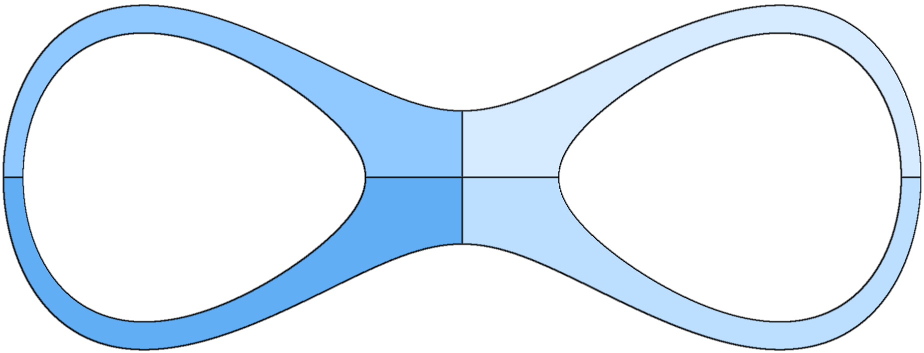
Example

Implicitly defined mesh — single phase



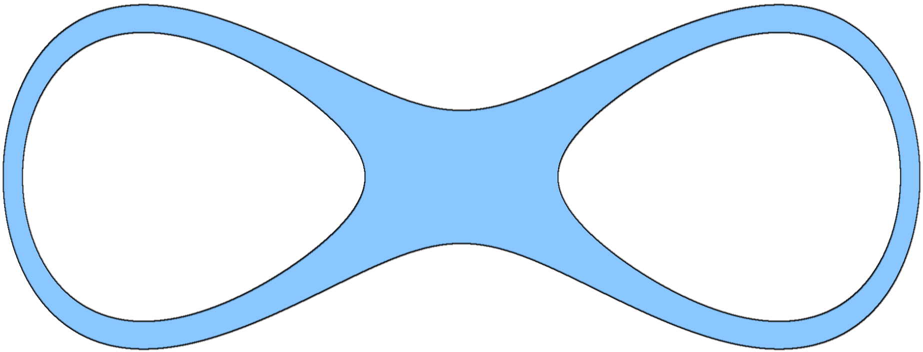
Example

Implicitly defined mesh — single phase



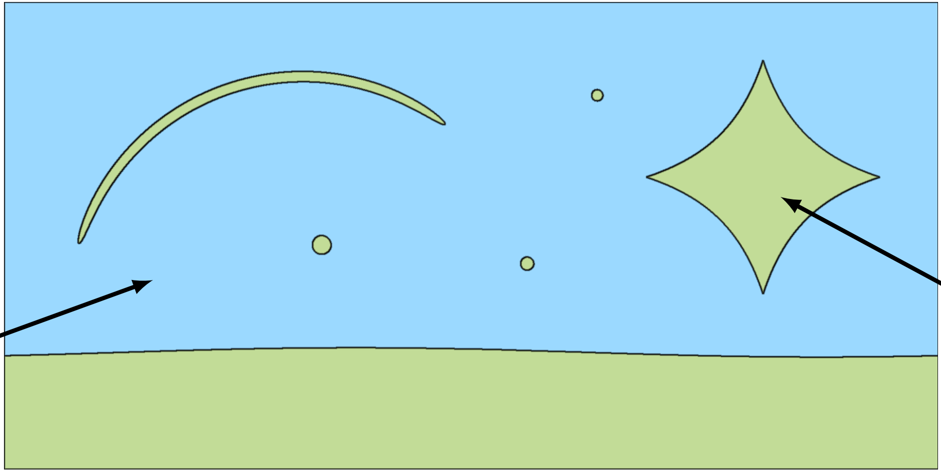
Example

Implicitly defined mesh — single phase



Example
Implicitly defined mesh — multiphase

$$-\nabla \cdot (\mu_i \nabla u) = 0, \quad [u]_\Gamma = [\mu_i \nabla u \cdot \boldsymbol{n}]_\Gamma = 0, \quad u|_{\partial\Omega} = 0$$

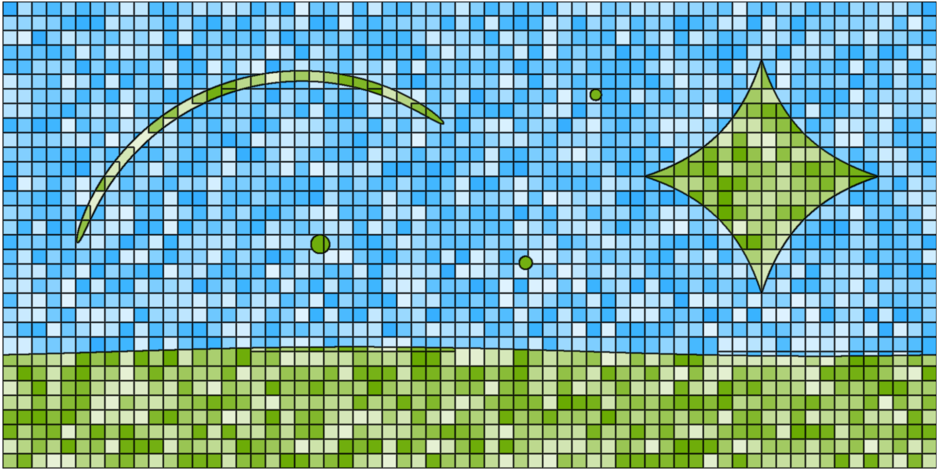


$\mu_2 = 4$

$\mu_1 = 1$

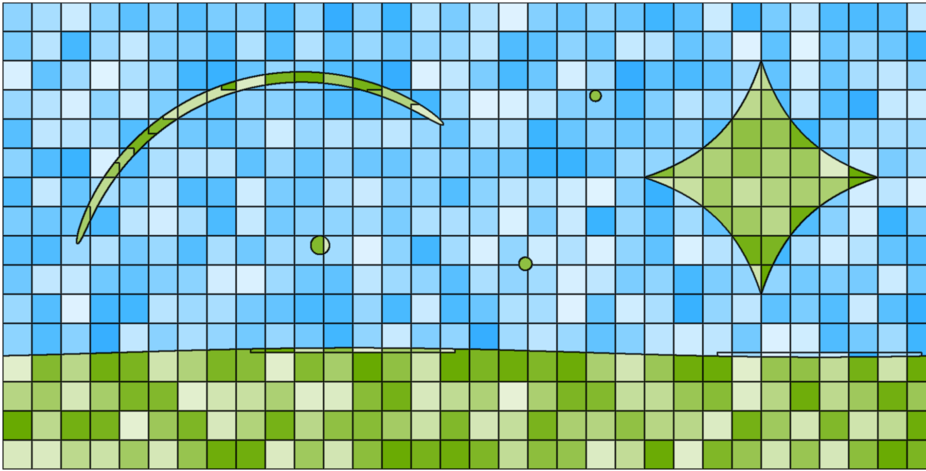
Example

Implicitly defined mesh — multiphase



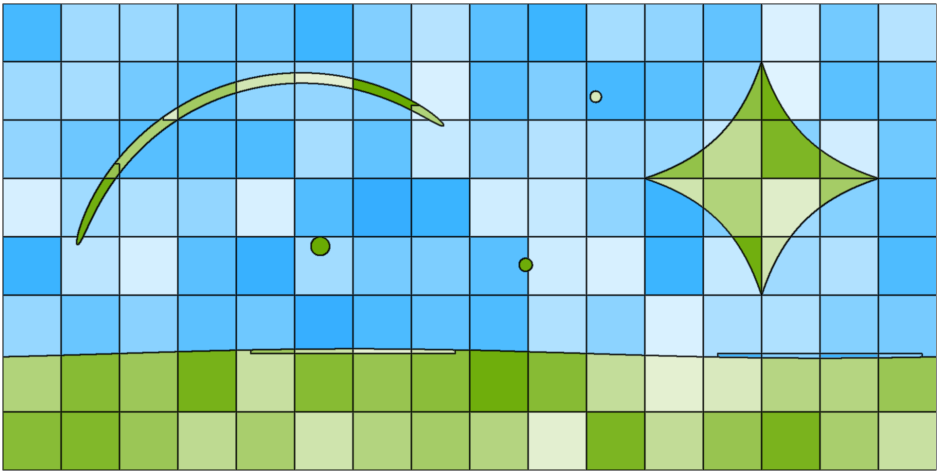
Example

Implicitly defined mesh — multiphase



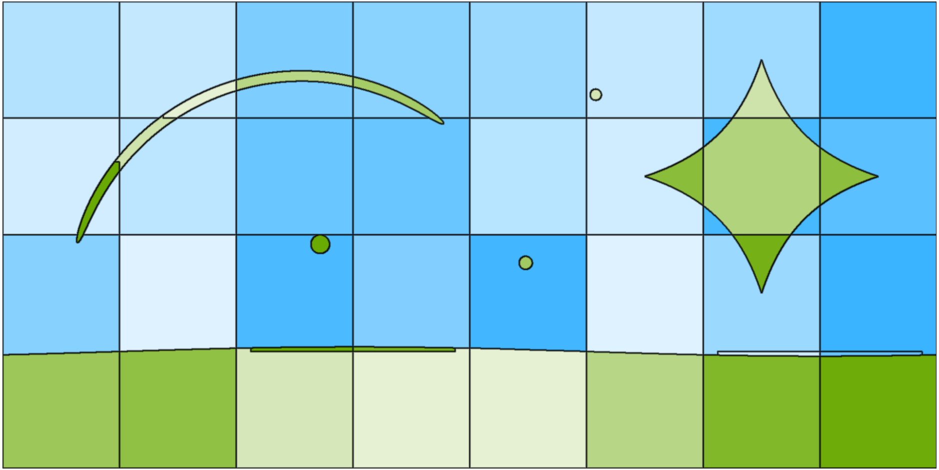
Example

Implicitly defined mesh — multiphase



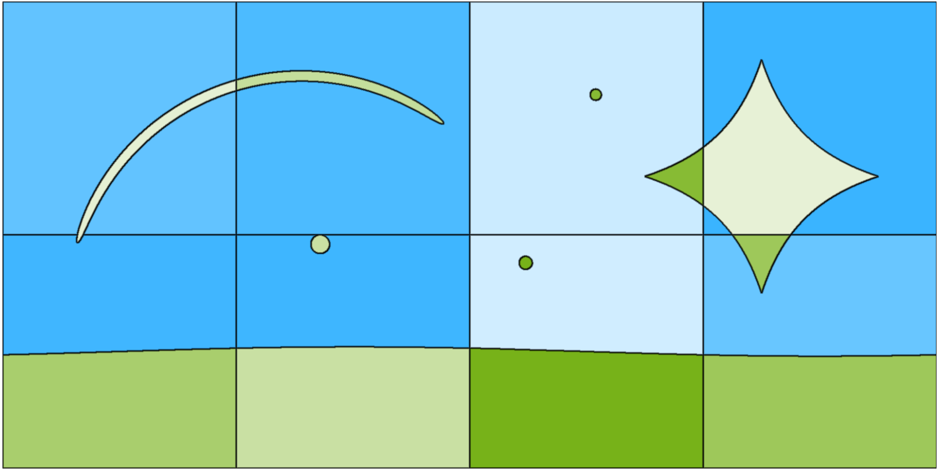
Example

Implicitly defined mesh — multiphase



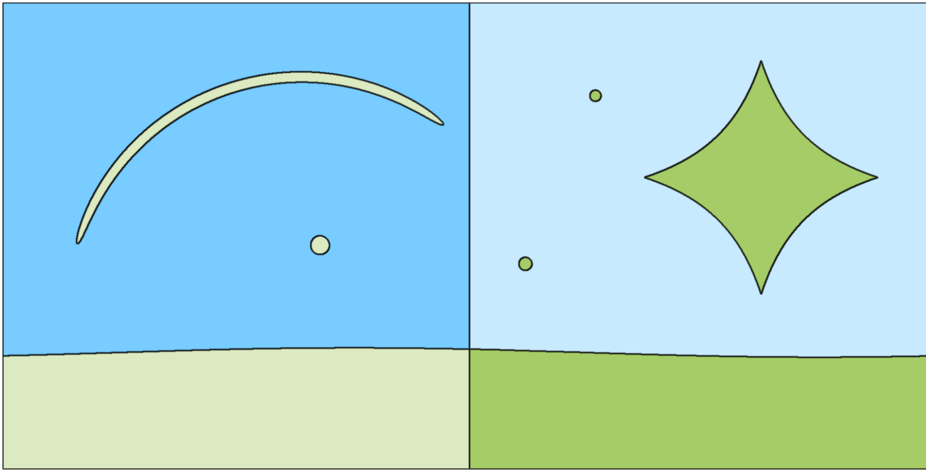
Example

Implicitly defined mesh — multiphase



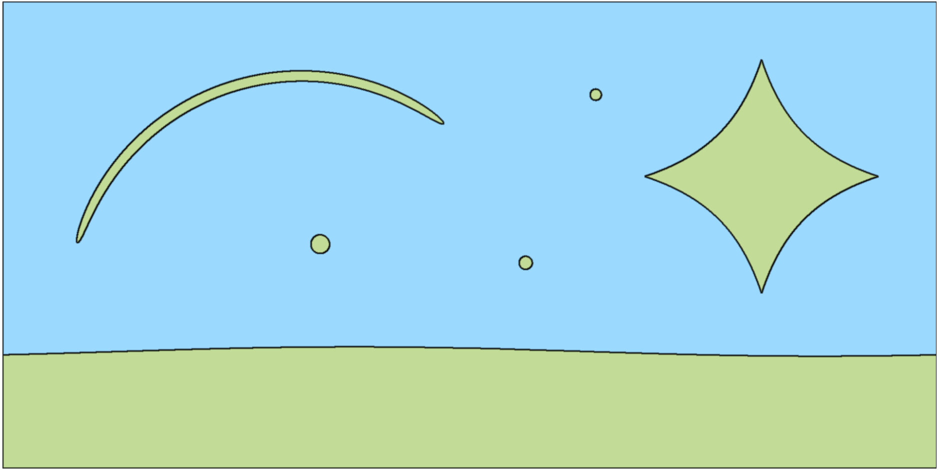
Example

Implicitly defined mesh — multiphase



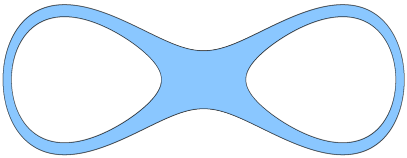
Example

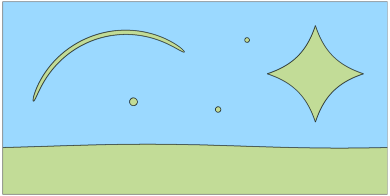
Implicitly defined mesh — multiphase

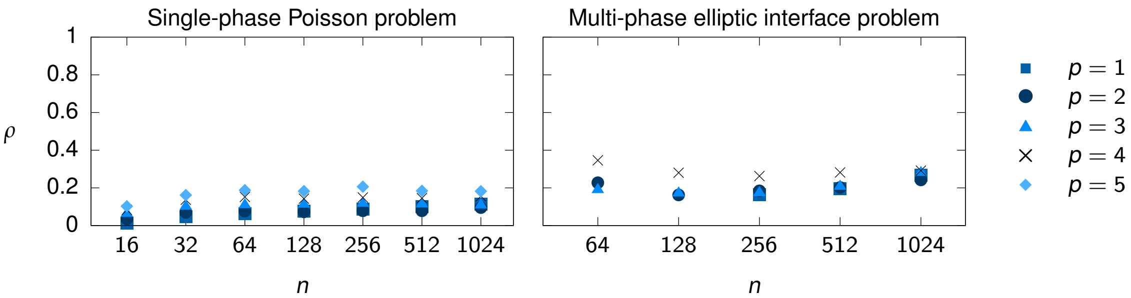


Example

Implicitly defined mesh — multiphase







Thanks for listening!

Preprint available: `arxiv.org/abs/1808.05320`