# The semi-Lagrangian method (for advection)

- A mix between purely Lagrangian (i.e. particle tracking) and purely Eulerian (i.e. fixed grid).
- Unconditionally stable, no CFL time-step constraint
  - ↳ Can take large time-steps
- Historically, used in numerical weather prediction for geophysical flows.

**Advection equation:** $\phi_t + \underline{u} \cdot \nabla \phi = 0$

Eulerian: CFL condition $\qquad \underline{u} = \underline{u}(\underline{x}, t)$

$$\frac{|u|\Delta t}{\Delta x} \le 1 \qquad \underbrace{\phantom{xxxxxx}}_{\text{material derivative}} \quad \frac{D}{Dt} := \frac{\partial}{\partial t} + \underline{u} \cdot \nabla$$

(B.E. stable but needs solve)

Lagrangian POV: $\frac{D\phi}{Dt} = 0, \quad \frac{D\xi}{Dt} = \underline{u}(\xi, t), \quad \xi(0) = \underline{x}.$

(clustering)

Semi-Lagrangian POV: $\frac{D\phi^{(k+1)}}{Dt} = 0$ over $[t_k, t_{k+1}]$ (1)

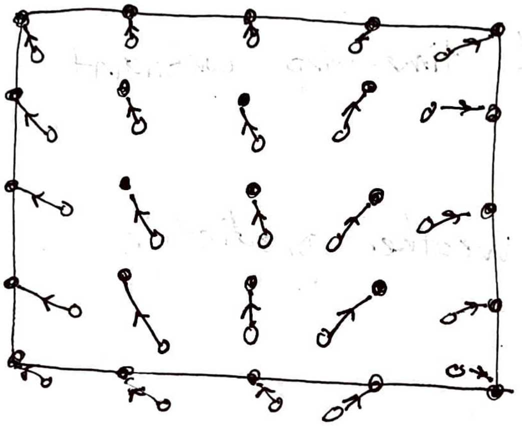
$$\frac{\phi^{(k+1)}(\xi) - \phi^{(k)}(\xi)}{\Delta t} = 0 \quad ? \qquad \frac{D\xi}{Dt} = \underline{u}, \quad \xi(t^{k+1}) = \underline{x} \qquad (2)$$

(no clustering)

Idea: We want to have an Eulerian grid at time $t^{k+1}$, i.e. $\{\underline{x}_j\}_{j=1}^{n}$. If we take the SL POV in $[t_k, t_{k+1}]$, where ~~Should~~ would $\underline{x}_j$ have started from (at $t_k$)? That is, find the departure points $\xi_j$ such that $\xi_j(t^{k+1}) = \underline{x}_j$.

(other direction is inverse NUFFT!)

To find $\xi_j$, solve (2) __backwards__ in time using an ODE integrator. There are n independent ODEs for each $\xi_j$. (Error: $O(\Delta t^k)$)



$\bullet = x_j$, know $\phi^{(k)}(x_j)$

$\circ = \xi_j$, become $\underbrace{\phi^{(k+1)}(x_j)}$

$\parallel$

$\phi^{(k)}(\xi_j)$

__Interpolation__

To interpolate $\phi_j^{(k)}$ from $x_j$ to $\xi_j$, using p-order accuracy. For spectral, this is a __NUFFT__. (Error: $O(\Delta x^{p+1})$).

__Overall error__: $$\frac{\phi^{(k+1)}(\xi) - \phi^{(k)}(\xi)}{\Delta t} = \frac{D\phi}{Dt} + O\left(\overset{1}{\Delta t^k} + \overset{2}{\frac{\Delta x^{p+1}}{\Delta t}}\right).$$

For low-order interp. (p small), 2 dominates, and increasing $\Delta t$ actually __decreases__ the error!

For high-order interp. (NUFFT), 1 dominates, increasing $\Delta t$ increases the error. Can take larger k to reduce.

The physical trajectory is always contained in the numerical domain of dependence (through interp.) so __no CFL condition__.